Physics 566: Quantum Optics I

Problem Set 2 Solutions

Problem 1:

(a) An arbitrary pure state of spin-½ $|\psi\rangle = \alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle = |\alpha|e^{i\phi_\alpha}|\uparrow_z\rangle + |\beta|e^{i\phi_\beta}|\downarrow_z\rangle$.
The overall phase is irrelevant, and $|\alpha|^2+|\beta|^2=1 \Rightarrow |\psi\rangle$ is specified by 2 real parameters:
$|\psi\rangle = |\alpha|\,|\uparrow_z\rangle + \sqrt{1-|\alpha|^2}\, e^{i(\phi_\beta-\phi_\alpha)}|\downarrow_z\rangle$. The eigenstate spin-up along the direction $\vec{e}_n$ can be written $|\uparrow_n\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$, where θ,ϕ define the direction on the sphere.

$$\boxed{\Rightarrow \forall |\psi\rangle \in \mathbb{C}^2 \quad |\psi\rangle \text{ is equivalent to } |\uparrow_n\rangle, \text{ with the association } |\alpha| = \cos\tfrac{\theta}{2},\ \phi_\beta - \phi_\alpha = \phi}$$

(b) Consider the projector

$$|\uparrow_n\rangle\langle\uparrow_n| = \left(\cos\tfrac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|\downarrow_z\rangle\right)\left(\cos\tfrac{\theta}{2}\langle\uparrow_z| + e^{-i\phi}\sin\tfrac{\theta}{2}\langle\downarrow_z|\right)$$

$$= \cos^2\tfrac{\theta}{2}|\uparrow_z\rangle\langle\uparrow_z| + \sin^2\tfrac{\theta}{2}|\downarrow_z\rangle\langle\downarrow_z| + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\left(|\uparrow_z\rangle\langle\downarrow_z|e^{i\phi} + |\downarrow_z\rangle\langle\uparrow_z|e^{i\phi}\right)$$

Using: $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$, $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$, $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$

$$\Rightarrow |\uparrow_n\rangle\langle\uparrow_n| = \tfrac{1}{2}\left(\hat{\mathbb{1}} + \cos\theta\,\hat{\sigma}_z + \sin\theta\left(e^{-i\phi}\hat{\sigma}_+ + e^{i\phi}\hat{\sigma}_-\right)\right) = \tfrac{1}{2}\left(\hat{\mathbb{1}} + \sin\theta(\cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y) + \cos\theta\,\hat{\sigma}_z\right)$$

$$\boxed{\Rightarrow |\uparrow_n\rangle\langle\uparrow_n| = \tfrac{1}{2}(\hat{\mathbb{1}} + \vec{e}_n\cdot\hat{\vec{\sigma}}), \text{ where } \vec{e}_n = \sin\theta(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \cos\theta\,\vec{e}_z}$$

(c) Consider $|\langle\uparrow_n|\uparrow_{n'}\rangle| = \left|\left(\cos\frac{\theta}{2}\langle\uparrow_z| + e^{-i\phi}\sin\frac{\theta}{2}\langle\downarrow_z|\right)\left(\cos\frac{\theta'}{2}|\uparrow_z\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|\downarrow_z\rangle\right)\right|$

$$= \left|\cos\tfrac{\theta}{2}\cos\tfrac{\theta'}{2} + e^{i(\phi'-\phi)}\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}\right| = \left[\left(\cos\tfrac{\theta}{2}\cos\tfrac{\theta'}{2} + \cos(\phi-\phi')\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}\right)^2 + \sin^2(\phi-\phi')\sin^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2}\right]^{1/2}$$

$$= \left[\cos^2\tfrac{\theta}{2}\cos^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2} + \tfrac{1}{2}\sin\theta\sin\theta'\cos(\phi-\phi')\right]^{1/2} = \left[\tfrac{1}{2}\left(1 + \cos\theta\cos\theta' + \sin\theta\sin\theta\cos(\phi-\phi')\right)\right]^{1/2}$$

$$= \left[\tfrac{1}{2}(1 + \vec{e}_n\cdot\vec{e}_{n'})\right]^{1/2} = \sqrt{\frac{1+\cos\Theta}{2}} \Rightarrow \boxed{|\langle\uparrow_n|\uparrow_{n'}\rangle| = \left|\cos\frac{\Theta}{2}\right|,\ \text{where } \cos\Theta = \vec{e}_n\cdot\vec{e}_{n'}}$$

Note: Antiparallel states $\Theta = \pi \Rightarrow |\langle\uparrow_n|\uparrow_{-n}\rangle| = \left|\cos\frac{\pi}{2}\right| = 0$ as expected

Aside: There is a more elegant solution to part (c) using the Trace operation

$$|\langle\Uparrow_n|\Uparrow_{n'}\rangle|^2 = \mathrm{Tr}\left(|\Uparrow_n\rangle\langle\Uparrow_n|\Uparrow_{n'}\rangle\langle\Uparrow_{n'}|\right) = \mathrm{Tr}\left[\left(\frac{\hat{\mathbb{1}}+\hat{\sigma}_n}{2}\right)\left(\frac{\hat{\mathbb{1}}+\hat{\sigma}_{n'}}{2}\right)\right] = \tfrac{1}{4}\mathrm{Tr}(\hat{\mathbb{1}}) + \tfrac{1}{4}\mathrm{Tr}(\hat{\sigma}_n) + \tfrac{1}{4}\mathrm{Tr}(\hat{\sigma}_{n'})$$
$$+ \tfrac{1}{4}\mathrm{Tr}(\hat{\sigma}_n\hat{\sigma}_{n'})]$$

$\mathrm{Tr}(\hat{\mathbb{1}})=2$, $\mathrm{Tr}(\hat{\sigma}_n)=\mathrm{Tr}(\hat{\sigma}_{n'})=0$, $\mathrm{Tr}(\hat{\sigma}_n\hat{\sigma}_{n'}) = \mathrm{Tr}(\vec{e}_n\cdot\hat{\vec{\sigma}}\hat{\vec{\sigma}}\cdot\vec{e}_{n'}) = \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j)(\vec{e}_i\cdot\vec{e}_n)(\vec{e}_j\cdot\vec{e}_{n'})$ Sum over $i+j$

$$= 2\,\delta_{ij}(\vec{e}_i\cdot\vec{e}_n)(\vec{e}_j\cdot\vec{e}_{n'}) = \vec{e}_n\cdot\vec{e}_{n'}$$

$$\Rightarrow |\langle\Uparrow_n|\Uparrow_{n'}\rangle|^2 = \tfrac{1}{2}(1+\vec{e}_n\cdot\vec{e}_{n'}) = \tfrac{1}{2}(1+\cos\Theta) = \frac{\cos^2\Theta}{2} \checkmark$$

(d) $\langle\Uparrow_n|\hat{\sigma}_i|\Uparrow_n\rangle = \mathrm{Tr}\left(|\Uparrow_n\rangle\langle\Uparrow_n|\hat{\sigma}_i\right) = \mathrm{Tr}\left(\left(\frac{\mathbb{1}+\hat{\sigma}_n}{2}\right)\hat{\sigma}_i\right) = \tfrac{1}{2}\mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_n) = \vec{e}_i\cdot\vec{e}_n$

$\Rightarrow \langle\Uparrow_n|\hat{\vec{\sigma}}|\Uparrow_n\rangle = \vec{e}_n$, the Bloch vector.

Problem 2: Spin Precession in a Magnetic Field (Heisenberg Picture)

$$\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B}, \quad \hat{\vec{\mu}} = -\gamma\hat{\vec{S}} \Rightarrow \hat{H} = \gamma\vec{B}\cdot\vec{S}$$

(a) Heisenberg equations of motion for i^{th} component of spin:

$$\frac{d}{dt}\hat{S}_i = \frac{-i}{\hbar}[\hat{S}_i, \hat{H}] = \frac{-i}{\hbar}\gamma B_j[\hat{S}_i, \hat{S}_j] = \left(\frac{-i}{\hbar}\gamma B_j\right)(i\hbar\epsilon_{ijk}\hat{S}_k) = \gamma\epsilon_{ijk}B_j\hat{S}_k \quad \left(\text{Einstein summation}\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\hat{\vec{S}} = \gamma\vec{B}\times\hat{\vec{S}} = \vec{\Omega}\times\hat{\vec{S}}}$$

Equation of a gyroscope

(b) $\hat{H} = \gamma\vec{B}\cdot\vec{S} = \frac{\hbar\gamma\vec{B}}{2}\cdot\hat{\vec{\sigma}} = \frac{\hbar\vec{\Omega}}{2}\cdot\hat{\vec{\sigma}} = \frac{\hbar}{2}(\Omega_-\hat{\sigma}_+ + \Omega_+\hat{\sigma}_- + \Omega_z\hat{\sigma}_z)$, $\quad \hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$

$\Omega_\pm = \Omega_x \pm \Omega_y$

Commutators: $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$, $\quad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$

$\Rightarrow$ Heisenberg eqns of motion: $\frac{d}{dt}\hat{\sigma}_z = -\frac{i}{\hbar}[\hat{\sigma}_z, \hat{H}] = -i(\Omega_-\hat{\sigma}_+ - \Omega_+\hat{\sigma}_-)$

$$\frac{d}{dt}\hat{\sigma}_- = -\frac{i}{\hbar}[\hat{\sigma}_+, \hat{H}] = \frac{-i}{2}\left([\hat{\sigma}_+, \hat{\sigma}_-]\Omega_+ + [\hat{\sigma}_+, \hat{\sigma}_z]\Omega_z\right) = -\frac{i}{2}\Omega_+\hat{\sigma}_z + i\Omega_z\hat{\sigma}_+$$

Note: In the standard spin-resonance problem, $\Omega_+ = \Omega_- = \Omega$, $\quad \Omega_z = -\Delta$

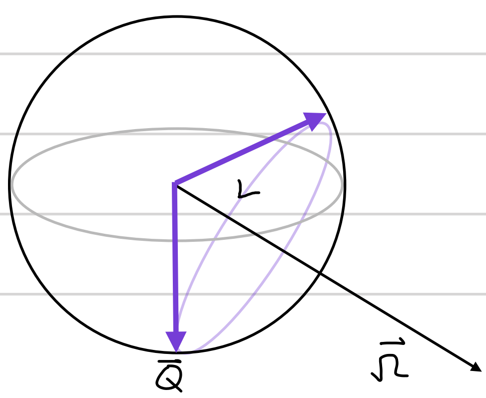
We recover the standard optical Bloch equations with $w = \langle\hat{\sigma}_z\rangle$, $\quad u - iv = \langle\hat{\sigma}_+\rangle$

(c) Solving for $\hat{\vec{S}}(t)$: Take a second time derivative (assuming static $\vec{B}$)

$$\frac{d^2}{dt^2}\hat{\vec{S}} = \vec{\Omega}\times\frac{d\hat{\vec{S}}}{dt} = \vec{\Omega}\times(\vec{\Omega}\times\hat{\vec{S}}) = \vec{\Omega}(\vec{\Omega}\cdot\hat{\vec{S}}) - \Omega^2\hat{\vec{S}} \Rightarrow \underbrace{\frac{d\hat{\vec{S}}_\perp}{dt^2} = -\Omega^2\hat{\vec{S}}_\perp}_{\text{Perpendicular component to } \vec{\Omega}}, \quad \frac{d\hat{S}_\parallel}{dt} = 0$$

$$\Rightarrow \hat{\vec{S}}_\perp(t) = \hat{\vec{S}}_\perp(0)\cos(\Omega t) + \frac{d\vec{S}_\perp(0)}{dt}\frac{1}{\Omega}\sin(\Omega t)$$

Trajectory of the Bloch vector



Problem 3

The two dimensional vector space the specifies the polarization state of a photon defines a qubit. We make the association:

$$\vec{e}_+ = \frac{\vec{e}_H + i\vec{e}_V}{\sqrt{2}} \Rightarrow |\uparrow_z\rangle$$

$$\vec{e}_- = \frac{\vec{e}_H - i\vec{e}_V}{\sqrt{2}} \Rightarrow |\downarrow_z\rangle$$

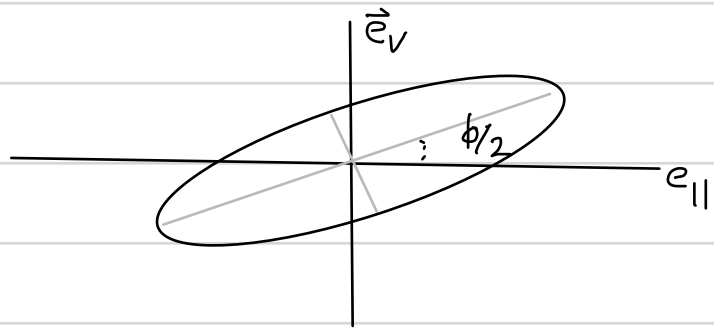
(a) $|\updownarrow_x\rangle = \frac{|\uparrow_z\rangle \pm |\downarrow_z\rangle}{\sqrt{2}} \Longleftrightarrow \frac{\vec{e}_+ \pm \vec{e}_-}{\sqrt{2}} \Rightarrow$ $\boxed{\begin{array}{l} |\uparrow_x\rangle \Leftrightarrow \vec{e}_H \text{ (linear horizontal)} \\ |\downarrow_x\rangle \Leftrightarrow i\vec{e}_V \equiv \vec{e}_V \text{ (linear vertical)} \end{array}}$

$|\updownarrow_y\rangle = \frac{|\uparrow_z\rangle \pm i|\downarrow_z\rangle}{\sqrt{2}} \Longleftrightarrow \frac{\vec{e}_+ \pm i\vec{e}_-}{\sqrt{2}} \Rightarrow$ $\boxed{\begin{array}{l} |\uparrow_y\rangle \Leftrightarrow \frac{1+i}{\sqrt{2}}\left(\frac{\vec{e}_H + \vec{e}_V}{\sqrt{2}}\right) \equiv \frac{\vec{e}_H + \vec{e}_V}{\sqrt{2}} \text{ (liner at } 45^\circ \text{ between } \vec{e}_H \text{ and } \vec{e}_V) \\ |\downarrow_y\rangle \Leftrightarrow \frac{1-i}{\sqrt{2}}\left(\frac{\vec{e}_H - \vec{e}_V}{\sqrt{2}}\right) \equiv \frac{\vec{e}_H - \vec{e}_V}{\sqrt{2}} \text{ (linear at } -45^\circ \text{ between } \vec{e}_H \text{ and } \vec{e}_V) \end{array}}$

(b) For an arbitrary state of the qubit, $|\uparrow_n\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$, where (θ, ϕ) is the direction on the Poincaré sphere.

$$\Rightarrow |\uparrow_n\rangle \equiv \cos\frac{\theta}{2}\,\vec{e}_+ + e^{i\phi}\sin\frac{\theta}{2}\,\vec{e}_- .$$

Recall (e.g. see Jackson 3rd edition Chap 7.2), the polarization is generally eliptical



$$r \equiv \frac{\alpha_+}{\alpha_-} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}e^{i\phi} = \cot\frac{\theta}{2}\,e^{i\phi}$$

Ratio of semimajor/semiminor axis $= \frac{1+r}{1-r} = \frac{1+\cot\frac{\theta}{2}}{1-\cot\theta/2} = \frac{1+\sin\theta}{1-\sin\theta}$

The ellipticity is characterized by $\left||\alpha_+|^2 - |\alpha_-|^2\right| = \left|\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right| = |\cos\theta|$

The orientation of the ellipse is shown, making an angle $\phi/2$ w.r.t. $\vec{e}_H$.

(c) Sketch of the Poincaré sphere

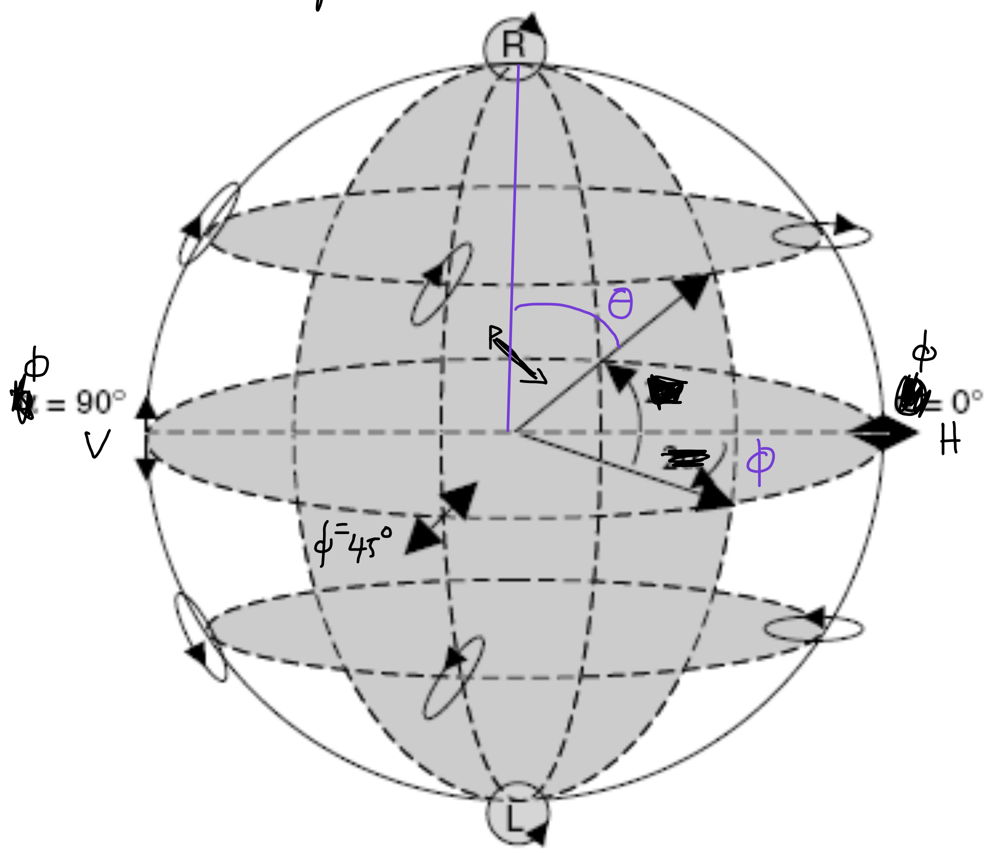


Figure 2.7. Poincaré sphere.

(d) Wave plate induces a phase shift that differs for "ordinary" and "extra ordinary" polarization. Thus, in the $\vec{e}_o$, $\vec{e}_e$ basis if

$$\vec{E}_{in} = \alpha_o \vec{e}_o + \alpha_e \vec{e}_e \Rightarrow \vec{E}_{out} = \alpha_o e^{i\phi_o} \vec{e}_o + \alpha_e e^{i\phi_e} \vec{e}_e$$

$$\doteq \begin{bmatrix} \alpha_o \\ \alpha_e \end{bmatrix} \qquad \doteq \underbrace{\begin{bmatrix} e^{i\phi_o} & 0 \\ 0 & e^{i\phi_e} \end{bmatrix}}_{U_{wp}} \begin{bmatrix} \alpha_o \\ \alpha_e \end{bmatrix}$$

U_{wp} is not in SU(2), because $\det[U_{wp}] = e^{i(\phi_e + \phi_o)}$. To make it in SU(2), divide by $\frac{1}{2}\det[U_{wp}]$

$$U_{wp} \Rightarrow \begin{bmatrix} e^{-i\Delta\phi/2} & 0 \\ 0 & e^{+i\Delta\phi/2} \end{bmatrix} \quad \text{where } \Delta\phi = \phi_e - \phi_o :$$

Recall, eigenvectors of SU(2) . $e^{\pm i\phi}$

We seek this relative to a basis of the Poincaré sphere. Note that

$\vec{e}_o = \cos\theta \vec{e}_H + \sin\theta \vec{e}_V$, $\quad \vec{e}_e = -\sin\theta \vec{e}_H + \cos\theta \vec{e}_V$ (rotation on Poincare sphere by 2θ around z.

$\Rightarrow$ Similarity transformation $\begin{bmatrix} \alpha_H \\ \alpha_V \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{S} \begin{bmatrix} \alpha_o \\ \alpha_e \end{bmatrix}$

$\Rightarrow$ In the basis $(\vec{e}_H, \vec{e}_V)$

$$U_{WP} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{-i\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{i\frac{\Delta\phi}{2}} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$\Rightarrow \boxed{U_{WP} = \begin{bmatrix} \cos\left(\frac{\Delta\phi}{2}\right) + i\cos 2\theta \sin\left(\frac{\Delta\phi}{2}\right) & -i\sin\left(\frac{\Delta\phi}{2}\right)\sin 2\theta \\ -i\sin\left(\frac{\Delta\phi}{2}\right)\sin 2\theta & \cos\left(\frac{\Delta\phi}{2}\right) - i\cos 2\theta \sin\left(\frac{\Delta\phi}{2}\right) \end{bmatrix}}$$

(c) A quarter-wave plate, $L = \frac{\lambda}{4(n_e - n_o)}$ $\Rightarrow$ $\Delta\phi = \frac{\pi}{2}$

$$\Rightarrow U_{QWP} = \begin{bmatrix} \frac{1 + i\cos 2\theta}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\sin 2\theta \\ -\frac{i}{\sqrt{2}}\sin 2\theta & \frac{1 - i\cos 2\theta}{\sqrt{2}} \end{bmatrix}$$

To tranform linear and $\vec{e}_H$ to $\vec{e}_+$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{-i}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

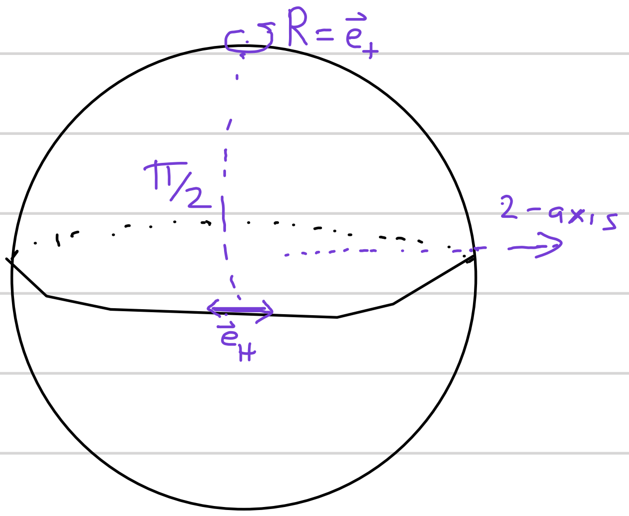
$\Rightarrow$ Choose $\theta = \frac{-\pi}{4}$ (45° between fast & slow axes) $\Rightarrow$

$U_{QWP}(\theta = \frac{\pi}{4}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ in $\vec{e}_H$ $\vec{e}_V$ basis $\Rightarrow$ Rotation by $-\frac{\pi}{2}$ around 2-axis of Poincaré

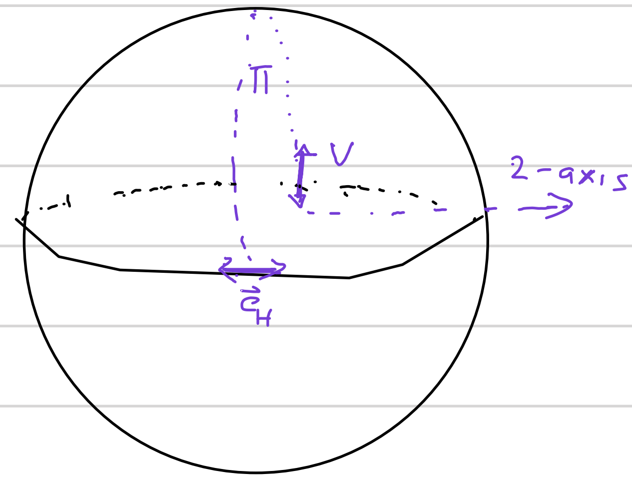
A half-wave plate, $L = \frac{\lambda}{2(n_e - n_o)}$ $\Rightarrow$ $\Delta\phi = \pi$

$$\Rightarrow U_{QWP} = -i\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

To transform $\vec{e}_H \Rightarrow \vec{e}_V$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ We achieve this by choosing $\theta = \frac{\pi}{4}$. The transformation is a rotation about 2-axis by π.



Quarter-wave plate



Half-wave plate

From this geometric construction we see immediately how to orient the wave plate. The axis of rotation (the eigenvector of the waveplate) should be half way between the $\vec{e}_H$ + $\vec{e}_V$ directions. The quarter wave plate then is a $\frac{\pi}{2}$ rotation on the Poincaré sphere, corresponding to $\vec{e}_H \Rightarrow \vec{e}_+$, $\vec{e}_V \Rightarrow \vec{e}_-$. The half-wave plate maps $\vec{e}_H \Rightarrow \vec{e}_V$, $\vec{e}_V \Rightarrow \vec{e}_H$ (overall phase irrelevant).

(f) From part (d), the SU(2) rotation corresponding to a wave plate, with the crystal axes oriented at angle θ w.r.t $\vec{e}_H, \vec{e}_V$ direction is $U^{WP}_\theta(\Delta\phi) = e^{-i\frac{\Delta\phi}{2}\vec{e}(\theta)\cdot\hat{\vec{\sigma}}} = \cos\frac{\Delta\phi}{2}\hat{1} - i(\cos 2\theta\,\hat{\sigma}_1 + \sin 2\theta\,\hat{\sigma}_2)\sin\frac{\Delta\phi}{2}$. (Note: A subtle point — in part (d) we wrote the matrices in the $\vec{e}_H, \vec{e}_V$ basis this defines $\hat{\sigma}_1$ in the usual Poincaré sphere) Thus, for a quarter waveplate and half waveplate respectively:

$$\text{QWP: } U^{WP}_\theta\left(\tfrac{\pi}{2}\right); \qquad \text{HWP: } U^{WP}_\theta(\pi)$$

We seek to show that we can construct an arbitrary SU(2) rotation on the Poincaré sphere with two QWP and one HWP. To do this I will employ the Euler angle parameterization. Recall

$$U \in SU(2) \Rightarrow \exists\, \alpha,\beta,\gamma \text{ (Euler angles) s.t. } U = U_3(\alpha)\,U_2(\beta)\,U_3(\gamma)$$ (rotation about 3-axis, then 2-axis, then 3-axis)

Now, note $U_3(2\theta)\,U_1(\frac{\Delta\phi}{2})\,U_3^\dagger(2\theta) = U^{WP}_\theta(\Delta\phi)$

Thus a sequence of QWP. HWP. QWP

$$\Rightarrow U^{WP}_{\theta_1}\left(\tfrac{\pi}{2}\right) U^{WP}_{\theta_2}(\pi)\, U^{WP}_{\theta_3}\left(\tfrac{\pi}{2}\right) = U_3(2\theta_1)\,U_1\left(\tfrac{\pi}{2}\right) U_3^\dagger(2\theta_1)\, U_3(2\theta_2)\, U_1(\pi)\, U_3^\dagger(2\theta_2)\, U_3(2\theta_3)\, U_1\left(\tfrac{\pi}{2}\right) U_3^\dagger(2\theta_3)$$

$$= U_3(2\theta_1)\,U_1\left(\tfrac{\pi}{2}\right) U_3(4\theta_2 - 2\theta_1 - 2\theta_3)\,U_1^\dagger\left(\tfrac{\pi}{2}\right) U_3(-2\theta_3) = U_3(2\theta_1)\,U_2(4\theta_2 - 2\theta_1 - 2\theta_3)\,U_3(-2\theta_3) = U_3(\alpha)\,U_2(\beta)\,U_3(\gamma)$$ Q.E.D.